A Multi-Agent Depth Bounded Boolean Logic*

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Abstract. Recent developments in the formalization of reasoning, especially in computational settings, have aimed at defining cognitive and resource bounds to express limited inferential abilities. This feature is emphasized by Depth Bounded Boolean Logics, an informational logic that models epistemic agents with inferential abilities bounded by the amount of virtual information they can use. However, such logics do not model the ability of agents to make use of information shared by other sources. The present paper provides a first account of a Multi-Agent Depth Bounded Boolean Logic, defining agents whose limited inferential abilities can be increased through a dynamic operation of becoming informed by other data sources.

Keywords: Logic of Information · Resource Bounded Reasoning · Information Transmission.

1 Introduction

Knowledge, belief and information are notions that have received increasing attention in the formal representation of cognition since the 60s, for their relevance to AI, distributed and multi-agent systems. Epistemic Logic (EL) [8, 14], with its interpretation of modal operators, has provided formal tools for dealing with knowledge and uncertainty from an agent-based perspective. Its Dynamic counterpart (DEL) [2, 9] has modelled change over knowledge bases, through the use of private and public announcement operations.

Informational logics aims at a similar task, offering formal interpretations for the epistemic states of being informed [1, 10, 11]; of holding the information [5–7]; and of becoming informed [15, 16]. While the former still interpret information states explicitly in the language through modal operators, the latter ones interpret propositional contents epistemically, in the vein of intuitionistic logic.

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In particular, Depth-Bounded Boolean Logic (DBBL) [3, 4, 6, 7] captures single-agent reasoning on the basis of informational states qualified as follows, see [13, p.177]:

1. **Informational truth** (T): agent j holds the information that ϕ is true;
2. **Informational falsehood** (F): agent j holds the information that ϕ is false;
3. **Informational indeterminacy** (*): agent j does not hold either the information that ϕ is true, or that it is false.

If bound to information actually held, agents reason at 0-depth. To resolve informational indeterminacy, an agent may use information not actually held. When a proposition ψ is undetermined, if the agent is able to derive it by supposing another formula ϕ to be true on the one hand, and false on the other, then ψ must hold irrespective of the information state concerning ϕ. In this way, agents are able to simulate informational states richer than the ones they actually are in, so as to infer further contents. For k-instances of such process, the agent is said to reason at k-depth. This informal concept is formalized both semantically and proof-theoretically. Each k-consequence and derivability relation characterizes a tractable logic, with classical logic obtained as the limit of their sequence.

DBBL models so far only the case in which virtual information is simulated by an agent's inner inferential abilities. In many practical applications, such information is communicated, as in DEL and in the epistemic operation of becoming informed in [16]. When in a context of informational indeterminacy, the cognitive process of an agent may be aided by becoming informed by another agent with stronger cognitive capabilities. For k-instances of such communication operation, the system may be said to implement multi-agent reasoning at k-depth. In the following, we propose MA-DBBL, a multi-agent version of DBBL, extending the latter with an operator of information sharing between agents, and one expressing information held by every agent in the system. This opens to multiple considerations on how to attribute such degrees of cognitive abilities to different agents, be they human or artificial. In a human-machine interaction setting, for example, cognitively bounded human agents may be aided in their process by mechanical ones. A measure of their inferential depth can be neutrally expressed as information quality (IQ, [12]), which provides qualitative parameters for evaluating data available for the inferential process.

The remaining of this paper is structured as follows. In Section 2 we introduce our logic and motivate the order on the agents by data quality criteria that justify inferential abilities. In Section 3 we provide the semantics, and in Section 4 the proof-theory for MA-DBBL. In Section 5 we provide soundness and completeness results. We conclude in Section 6 highlighting limitations and further possible extensions of the present system.

2 Multi-Agent DBBL

In Depth-Bounded Boolean logic, agents are characterised by two aspects: first, the information they currently hold as true, false or indeterminate; second, their inferential abilities. The latter aspect is typical of DBBL, considered especially in order to distinguish tractable fragments of classical logic. We consider a situation in which different epistemic agents are ordered by such characteristics. While current information held is a static property, the inferential abilities are usually linked to cognitive or computational capacities which allow to extend the information base of the agent. In the following, the inferential ability of an agent corresponds to a semantic interpretation of the data available to her. The result is a hierarchy of agents in which at each level the agent holds more information and also has better inferential abilities than the agent below.

Definition 1 (Syntax of MA-DBBL).

$$\mathcal{S} := \{i \preceq j \preceq k \preceq \cdots \preceq z\}$$

$$\mathcal{P} := \{\phi_i, \psi_i, \rho_i, \dots, \}$$

$$\mathcal{D} := \{Ac, Un, Us\}$$

$$\mathcal{C} := \{\wedge, \vee, \rightarrow, \neg\}$$

$$\mathcal{K} := \{BI_i, I_i\}$$

$\mathcal{S}$ is a finite set of agents where each agent is identified with the information she holds. A total preorder $\preceq$ is imposed on $\mathcal{S}$, such that $i \preceq j$ means that agent i holds at least as much information as agent j. This is a preference relation, i.e. it is reflexive ($\forall i \in \mathcal{S}, i \preceq i$); connexive ($\forall i, j \in \mathcal{S}, i \preceq j$ or $j \preceq i$); and transitive ($\forall h, i, j \in \mathcal{S}$ if $h \preceq i$ and $i \preceq j$, then $h \preceq j$). This guarantees that any two agents are comparable, and when considered equivalent in the hierarchy they have the same information. $\mathcal{P}$ is an enumerable set of propositional variables denoting data available to agents in $\mathcal{S}$, although not every agent has a determinate truth value for every such data. Propositional variables are indexed by agents, and closed under a set of propositional connectives $\mathcal{C}$; and a set of indexed epistemic operators $\mathcal{K}$. In order to define the preorder $\preceq$ and give a precise interpretation of the relation underlying it, we use elements of the set $\mathcal{D}$ over elements of the set $\mathcal{P}$. Note that it is essential to have $\mathcal{P}$ accessible in principle to all agents in order them to be ordered with respect to one another. In short, elements in $\mathcal{D}$ determine for each agent i which data is accessible, understandable and usable. Formally, these are functions on formulas or sets of formulas with the following meaning:[3]

$Ac(\phi_i) \leftrightarrow v(\phi_i) \in \{1, 0\}$
$Un(\phi_i) \leftrightarrow Ac(\psi_i)$, for some ψ_i such that ψ_i implies ϕ_i
$Us(\phi_i) \leftrightarrow Ac(\psi_i), \forall \psi_i$ implied by ϕ_i

[3] We use here a two-value valuation function v on formulas as a mapping to truth and falsity. Its formal definition, extended to a three-valued function, is postponed to Section 3.

The first function expresses the condition that information is accessible if the agent holds it true or false: this is a monotonic function, i.e. if the cardinality of the set of formulas Γ_i accessible to agent i is greater than the cardinality of the set of formulas Γ_j accessible to agent j then there is at least one formula ϕ_i which is accessible to i, but which is not accessible to j, formally $\mid Ac(\Gamma_i) \mid > \mid Ac(\Gamma_j) \mid \leftrightarrow \exists \phi_i \in Ac(\Gamma_i) \wedge \phi \notin Ac(\Gamma_j)$. In other words, all agents are characterized by a common database represented by $\mathcal{P}$, but each agent can have access to different elements of $\mathcal{P}$, depending on the truth values held for those elements. We shall see in Section 3 that the semantics of DBBL is three-valued and non-deterministic, hence accessible information is a subset of all information available to agents. The second function expresses the condition that information is understandable if the agent can access it from other accessible information. This, informally, means that understandable information is such when it is recognised as a consequence of accessible information. Finally, the third function expresses the condition that information is usable if it allows to access other information. This, informally, means that information is qualified as usable when it allows to perform inferences that make other information accessible. Accessibility is a monotonic function, hence agents higher in the hierarchy have access to an increasing set of formulas; so are therefore also the set of formulas that can be understood and used by them. We shall see in Section 4 how usability is the criterion which underlies the inferential depth of agents. Let us for now consider an informal example.[4]

Example 1. Suppose that agent j doesn't hold information about whether the sentence ϕ: "C is father of both A and B" is true or false, nor is she able to derive it by herself. Then ϕ is not accessible to agent j i.e. $\phi \notin Ac(\phi_j)$. Suppose also that agent i has not access to ϕ, that is $\phi \notin Ac(\phi_i)$, but she holds true the information ψ: "C is not father of A" and true the information χ: "C is not father of B". Then agent i has access to both ψ and χ i.e. $Ac(\psi_i, \chi_i)$. Moreover, agent i holds true the information ξ: "C is not father of A and C is not father of B". Then we obtain that ψ and χ are understandable to agent i, written $Un(\psi_i, \chi_i)$, for i is able to use those propositional contents in order to access new information. Finally, if agent i is able to establish that ϕ is false, then we conclude that $Us(\psi_i, \chi_i)$ and $Ac(\phi_i)$.

Information is usable if understood and accessible. In turn, given a shared information set, not every agent might have the same accessible, understandable or usable information. Information which is not accessible, understandable or usable denotes contents for which a truth value cannot be determined, or whose consequences cannot be inferred, or which cannot be inferred from other information. We then link the inferential ability of an agent i to her accessible, understandable and usable information. We define therefore our total preorder as follows:

Definition 2 (Source Order by Information Usability).

$$i \preceq j \ \textit{iff} \ \mid Us(\Gamma_i) \mid \geq \mid Us(\Gamma_j) \mid$$

[4] This example is based on an unpublished one formulated by Marcello D'Agostino.

Where Γ_i and Γ_j are set of formulas, and $\mid Us(\Gamma_i) \mid$ represents the cardinality of the set of formulas usable to agent i. Hence $i \preceq j$ holds if agent i has more information than j with a determined truth-value, from which she can infer new information and which she can infer from determined information.

Example 2. Take $\mathcal{S} = \{h, i, j\}$, $Ac(p_i \wedge q_i)$, $Un(p_i)$, $Us(p_i)$ so that $Ac(p_i \vee r_i)$. Now suppose $Ac(\{\}_j)$, it also holds $Un(\{\}_j)$ and $Us(\{\}_j)$. Hence $i \preceq j$. Moving to agent h, $Ac(r_h \rightarrow t_h)$, $Un(p_h)$ $Un(t_h)$, $Us(p_h)$ so that $Ac(p_h \vee r_h)$ and $Us(t_h)$ so that $Ac(t_h \wedge s_h)$. Now since $\mid Us(p_h, t_h) \mid \leq \mid Us(p_i) \mid$, $h \preceq i$ is the case, and given transitivity of $\preceq$ the order among agents is established.

Our goal is now to model reasoning in a context in which agents higher in the hierarchy can communicate information which is not directly accessible to those below. From now on, we say that information is held by an agent always meaning that it is usable by that agent in the sense defined above.

Definition 3 (Language).

$$\mathcal{L}^{\mathcal{CK}} := \{p_j | \phi_j \wedge \psi_j | \phi_j \vee \psi_j | \phi_j \rightarrow \psi_j \mid \neg\phi_j \mid BI_j\phi_i \mid I_j\phi_i\}$$

Given p_j denoting an indexed atomic variable, a metavariable for a formula ϕ_j denotes information ϕ held – i.e. usable – by an agent j. The set of information held by agent j is denoted as Γ_j for a set of formulae $\{\phi_j, \dots, \psi_j\}$. Formulas are closed under standard connectives. We use brackets to enclose composed formulas and use the index after the closing bracket, while avoiding brackets in the case of negated formulas for aiding readability. We refer to the non-epistemic fragment of our language as $\mathcal{L}^{\mathcal{C}}$, and the corresponding logic as MA-DBBL$^{\mathcal{C}}$. We refer to the epistemic fragment of our language as $\mathcal{L}^{\mathcal{K}}$, and the corresponding logic as MA-DBBL$^{\mathcal{K}}$. The latter is obtained from the former by adding two epistemic formulae with the indexed operators:

- $BI_j\phi_i$ says that "agent j becomes informed that ϕ by an agent $i \preceq j$";
- $I_j\phi_i$ says that "agent j is informed that ϕ by all agents $i \preceq j$".

$BI_j\phi_i$ simulates a private announcement of ϕ received by agent j from agent i. $I_j\phi_i$ expresses consensus on ϕ, similar in meaning to the epistemic operator "everybody knows".[5]

3 Semantics

MA-DBBL$^{\mathcal{C}}$ has a three-valued semantics which formalizes reasoning by an agent based on her actually held information, captured as evaluation of $\mathcal{L}^{\mathcal{C}}$ formulas.

[5] The two operators reflect the distinction between an alethically neutral and a veridical conception of information, see [16] and [10] respectively. Technically, it is possible to reformulate the present monotonic version of MA-DBBL without the I operator without loss of expressiveness. We keep it both in the language to preserve the mentioned conceptual distinction, and because it offers the basis for a planned extension of the present system with contradictory information updates.

$\wedge$	1_i	0_i	$*_i$
1_i	1_i	0_i	$*_i$
0_i	0_i	0_i	0_i
$*_i$	$*_i$	0_i	$0_i, *_i$

$\vee$	1_i	0_i	$*_i$
1_i	1_i	1_i	1_i
0_i	1_i	0_i	$*_i$
$*_i$	1_i	$*_i$	$1_i, *_i$

$\rightarrow$	1_i	0_i	$*_i$
1_i	1_i	0_i	$*_i$
0_i	1_i	1_i	1_i
$*_i$	1_i	$*_i$	$1_i, *_i$

$\neg$	
1_i	0_i
0_i	1_i
$*_i$	$*_i$

Table 1: Non-deterministic 3-valued truth tables for agent i

The matrices exposed in Table 1 express the informational meaning of the logical operators for this fragment, see [5]. We include values for information that agent i holds as true (1_i), information that agent i holds as false (0_i), and information which agent i cannot establish whether it is true or false ($*_i$). For formulas with determined values (i.e. where $*$ is not present), the tables are the classical ones. When only one element is undetermined, those tables work exactly as their classical counterparts: for example, the truth value of a conjunction is undetermined only when the element that is not undetermined is true, and false otherwise. But when considering, e.g. the conjunction of two undetermined elements, the resemblance with the classical tables falls apart. In such instances, the conjunction of two undetermined elements can have two different outputs: it could be false or undetermined. Which is the case, does not depend merely on the truth values of the components of the conjunction, but it also depends on the background information possessed by the agent. Example 1 from the previous section has already shown a case for this unusual behaviour.

Formula valuation for $\mathcal{L}^{\mathcal{C}}$ is defined as follows:

Definition 4 (Valuation of $\mathcal{L}^{\mathcal{C}}$ formulas). *A 3ND-valuation is a mapping $v : \Gamma_i \rightarrow \{1, 0, *\}^{\mathcal{L}^{\mathcal{C}}}$, satisfying the following conditions for all $\phi_i, \psi_i \in \Gamma_i$:*

1. $v(\neg\phi_i) = \mathcal{F}_{\neg}(v(\phi_i))$
2. $v(\phi_i \circ \psi_i) \in \mathcal{F}_{\circ}(v(\phi_i), v(\psi_i))$

where

(i) *$\mathcal{F}_{\neg}$ is the deterministic truth-function defined by 3-valued table for $\neg$, and*
(ii) *$\mathcal{F}_{\circ}$ is the non-deterministic truth-function defined by the 3-valued table for $\circ = \{\wedge, \vee, \rightarrow\}$.*

This valuation defines therefore the way in which an agent establishes truth, falsity and indeterminacy for the information she holds, without any aid offered by other sources. The consequence relation for the fragment $\mathcal{L}^{\mathcal{C}}$ is dubbed 0-depth and it is defined as follows:

Definition 5 **(0-depth consequence relation).** *For every set of formulas Γ_i and formula ϕ_i of $\mathcal{L}^{\mathcal{C}}$, we say that ϕ_i is a 0-depth consequence of Γ_i, denoted by $\Gamma_i \vDash_0 \phi_i$, if $v(\phi_i) = 1$ for every v as of Definition 4 such that $v(\psi_i) = 1, \forall \psi_i \in \Gamma_i$.*

Accordingly, a notion of inconsistency for $\mathcal{L}^{\mathcal{C}}$ holds as follows:

Definition 6 **(0-depth inconsistency).** *A set of formulas Γ_i of $\mathcal{L}^{\mathcal{C}}$ is inconsistent, denoted by $\Gamma_i \vDash_0 \bot$, if no valuation exists such that all formulas $\phi_i \in \Gamma_i$ are satisfied.*

When adding the BI and I operators, we move to the fragment $\mathcal{L}^{\mathcal{K}}$ of the language, with formulas indexed by multiple agents.

Definition 7 **(Valuation of $\mathcal{L}^{\mathcal{K}}$ formulas).**

A 3ND-valuation is a mapping $v : \Gamma_j \rightarrow \{1, 0, *\}^{\mathcal{L}^{\mathcal{K}}}$, satisfying the following conditions for all $\phi_j \in \Gamma_j$, and $\circ = \{1, 0, *\}$:

- $v(BI_j\phi_i) = \circ$ *iff* $v(\phi_i) = \circ$
- $v(I_j\phi_i) = \circ$ *iff* $v(BI_j\phi_i) = \circ$ *for all* $i \preceq j$.

Example 3. Suppose that agent j doesn't hold information about whether the sentence ϕ: "C is father of both A and B" is true or false, nor is she able to derive it by herself. Agent i, despite not holding a truth value for ϕ, is able to use the information that "C is not father of A" and that "C is not father of B", thus concluding that "C is father of both A and B" is false. Hence, agent j becomes informed that "C is father of both A and B" is false from agent i.

Example 4. Suppose $h \preceq j$ and $i \preceq j$. Suppose that agent j doesn't hold information about whether ϕ: "C is father of both A and B" is true or false, nor is she able to derive it by herself. Agent i, despite not holding a truth value for ϕ, is able to use the information that "C is not father of A" and that "C is not father of B", thus concluding that "C is father of both A and B" is false. Assume moreover that agent h is able to infer that "C is father of both A and B" is false, because she knows that "C is father of D" and "D is sibling of A and not sibling of B" are both true. Hence, agent j becomes informed that "C is father of both A and B" is false from agent i and from agent h separately. Then we say that j is informed that "C is father of both A and B" is false.

The notion of consequence relation for MA-DBBL expresses the principle that given $i \preceq j$, an agent j either establishes the truth of some formula ϕ by using the information she holds – i.e. consequence relation at 0-depth; or else, she becomes informed of a formula ϕ from agent i who is able to infer it on her own by using some additional information, or what are called virtual assumptions: the latter is a consequence relation at $k > 0$-depth. In order to bound the number of formulas that are allowed to be used as virtual assumptions up to a certain fixed depth, we introduce the notion of "virtual space", see [7, p.84]. Given any set Δ_i of formulae of $\mathcal{L}^{\mathcal{CK}}$; the function $Sub(\Delta_i)$ that returns the set of subformulae of Δ_i; and the function $At(\Delta_i)$ that returns the set of all its atomic subformulae; consider all operations f such that:

1. for all Δ_i, $At(\Delta_i) \subseteq f(\Delta_i)$
2. $Sub(f(\Delta_i)) = f(\Delta_i)$
3. $| f(\Delta_i) | \leq p(| \Delta_i |)$, for some fixed polynomial p and $| \Delta_i |$ denoting the size of Δ_i, i.e. the number of occurrences of symbols in that set.

The first criteria ensures that for every set of formulas Δ_i, the set of its atomic subformulae is contained in the set resulting from applying f to Δ_i; the second states that $f(\Delta_i)$ is closed under subformulae; the third ensures that any function f is at most polynomial in some size. While $Sub(\Delta_i)$ and $At(\Delta_i)$ are examples of such operations, in fact all operations that are built from them and have bounded logical complexity satisfy the above criteria. Our virtual space can now be defined as a function f of the set $\Gamma_i \cup \{\phi_i\}$ made of premises Γ_i and conclusion ϕ_i of the given inference for agent i.

Definition 8 (Consequence relation for MA-DBBL). *For every set of formulas Γ_j, formula ϕ_i of $\mathcal{L}^{\mathcal{C}}$, formulas $BI_j\phi_i$ and $I_j\phi$ of $\mathcal{L}^{\mathcal{K}}$, all operations f satisfying points 1-3 above and $i \preceq j$:*

- *$\Gamma_j \vDash_0^f \phi_i$ iff $\Gamma_j \vDash_0 \phi_i$;*
- *$\Gamma_j \vDash_{k+1}^f BI_j\phi_i$ iff $\Gamma_i \cup \psi_i \vDash_k^f \phi_i$ and $\Gamma_i \cup \neg\psi_i \vDash_k^f \phi_i$ for some $\psi_i \in f(\Gamma_i \cup \{\phi_i\})$ and some $i \preceq j$;*
- *$\Gamma_j \vDash_{k+1}^f I_j\phi$ iff $\Gamma_i \cup \psi_i \vDash_k^f \phi_i$ and $\Gamma_i \cup \neg\psi_i \vDash_k^f \phi_i$ for some $\psi_i \in f(\Gamma_i \cup \{\phi_i\})$ and all $i \preceq j$.*

This definition follows from Definitions 5 and 7, together with the function f defined above. The first case holds trivially, as f is irrelevant. The second case is shown to hold by reasoning by induction on the depth of k. Suppose that $k = 0$ and that $\Gamma_j \vDash_{k+1}^f BI_j\phi_i$ is valid, but that $\Gamma_i \cup \psi_i \nvDash_k^f \phi_i$ and $\Gamma_i \cup \neg\psi_i \nvDash_k^f \phi_i$ for some $\psi_i \in f(\Gamma_i \cup \{\phi_i\})$. Take $\Gamma_i \cup \psi_i \nvDash_k^f \phi_i$. As said, with $k = 0$ we have $\Gamma_i \cup \psi_i \nvDash_0^f \phi_i$. Hence, by Definition 4, $v(\phi_i) = 0$ and $v(\Gamma_i \cup \psi_i) = 1$. Now by Definition 7, if $v(\phi_i) = 0$ then $v(BI_j\phi_i) = 0$, which (by monotonicity of Us_i) means that $\Gamma_j \nvDash_{k+1}^f BI_j\phi_i$, against the assumption. The case for $k > 0$ is straightforward and the case for I formulas only generalises to all agents.

Definition 9 (Inconsistency for MA-DBBL). *A set of formulas Γ_i of $\mathcal{L}^{\mathcal{CK}}$ is k-depth inconsistent if and only if $\Gamma_i \cup \{\psi_i\}$ and $\Gamma_i \cup \{\neg\psi_i\}$ are both (k-1)-depth inconsistent for some $\psi_i \in f(\Gamma_i)$.*

Since $\vDash_0$ is monotonic, $\vDash_k^f \subseteq \vDash_{k+n}^f$, and the increase in depth corresponds to the use of additional virtual information by more agents, restricted by the space defined by f. Note also that we can define a partial order among operations in f, such that $f_1 \trianglelefteq f_2$ iff $f_1(\Delta_i) \subseteq f_2(\Delta_i)$, for every Δ_i. Hence $\vDash_k^{f_1} \subseteq \vDash_k^{f_2}$ whenever $f_1 \trianglelefteq f_2$. Then (see [7, p.85]):

Proposition 1. *The relation $\vDash_\infty^f = \bigcup_{k \in N} \vDash_k^f$ is the consequence relation of classical propositional logic*

$$\frac{\phi_i}{(\phi \vee \psi)_i} \vee - intro1 \qquad \frac{\psi_i}{(\phi \vee \psi)_i} \vee - intro2 \qquad \frac{\neg\phi_i \qquad \neg\psi_i}{\neg(\phi \vee \psi)_i} \vee - intro3$$

$$\frac{\phi_i \qquad \psi_i}{(\phi \wedge \psi)_i} \wedge - intro1 \qquad \frac{\neg\phi_i}{\neg(\phi \wedge \psi)_i} \wedge - intro2 \quad \frac{\neg\psi_i}{\neg(\phi \wedge \psi)_i} \wedge - intro3$$

$$\frac{\neg\phi_i}{(\phi \to \psi)_i} \to -intro1 \qquad \frac{\psi_i}{(\phi \to \psi)_i} \to -intro2 \qquad \frac{\phi_i \qquad \neg\psi_i}{\neg(\phi \to \psi)_i} \to -intro3$$

$$\frac{(\phi \wedge \psi)_i}{\phi_i} \wedge - elim1 \qquad \frac{(\phi \wedge \psi)_i}{\psi_i} \wedge - elim2 \qquad \frac{\neg(\phi \wedge \psi)_i \qquad \phi_i}{\neg\psi_i} \wedge - elim3$$

$$\frac{\neg(\phi \wedge \psi)_i \qquad \psi_i}{\neg\phi_i} \wedge - elim4$$

$$\frac{(\phi \vee \psi)_i \qquad \neg\phi_i}{\psi_i} \vee - elim1 \qquad \frac{(\phi \vee \psi)_i \qquad \neg\psi_i}{\phi_i} \vee - elim2$$

$$\frac{\neg(\phi \vee \psi)_i}{\neg\psi_i \quad \neg\phi_i} \vee - elim3$$

$$\frac{(\phi \to \psi)_i \qquad \phi_i}{\psi_i} \to -elim1 \qquad \frac{(\phi \to \psi)_i \qquad \neg\psi_i}{\neg\phi_i} \to -elim2$$

$$\frac{\neg(\phi \to \psi)_i}{\phi_i} \to -elim3 \qquad \frac{\neg(\phi \to \psi)_i}{\neg\psi_i} \to -elim4$$

Fig. 1: Introduction and Elimination Rules for MA-DBBL$^{\mathcal{C}}$

If we constrain one instance of a BI-formula for each distinct agent, higher depth expresses nesting of information requests from distinct sources, which means we obtain the full classical propositional logic only with an infinite number of agents.

4 Proof-Theory

To illustrate the proof-theory of MA-DBBL, we start again by separating the inferential abilities of the agent reasoning only on the basis of her actually held information, from her inferential abilities supported by information received from agents with a larger set of usable information. Introduction and elimination rules for MA-DBBL$^{\mathcal{C}}$ capture the idea of manipulating actual information, expressing inferential ability at 0-depth, see Figure 1.

Definition 10 (Derivability for MA-DBBL$^{\mathcal{C}}$). *Given a set of formulas Γ_i of $\mathcal{L}^{\mathcal{C}}$:*

- *An intelim sequence for Γ_i is a sequence $\phi_i, ..., \psi_i$ such that*
  - *each formula ϕ_i in the sequence is a member of Γ_i, or*
  - *is the conclusion of the application of a MA-DBBL$^{\mathcal{C}}$ rule;*
- *An intelim proof of ϕ_i from Γ_i is such that*
  - *either it is a closed sequence, i.e. it contains both ϕ_i and $\neg\phi_i$;*
  - *or ϕ_i is the last formula in the sequence;*
- *A formula ϕ_i is 0-depth derivable from Γ_i, written $\Gamma_i \vdash_0 \phi_i$, if and only if there is an intelim proof of ϕ_i from Γ_i according to MA-DBBL$^{\mathcal{C}}$ rules.*

Definition 11 (Refutability for MA-DBBL$^{\mathcal{C}}$). *Given a set of formulas Γ_i of $\mathcal{L}^{\mathcal{C}}$:*

- *An intelim refutation of Γ_i is a closed intelim sequence for Γ_i;*
- *Γ_i is intelim-refutable if there is a closed intelim sequence for Γ_i.*

Reasoning at 0-depth proves to be a weighty limit for an agent, for she cannot make any supposition that exceeds the information she actually holds, bounding her inferential abilities. The next step consists in finding a way to manipulate virtual information: when an agent is not able to infer the truth value of a formula ϕ by the information she holds, she can be informed about ϕ by an agent with higher inferential abilities. The rules of Figure 2 describe operations in which agent j receives information from one or several agents with access to more information. By convention the formula in the conclusion of a rule with multiple premises always carries the index of the highest agent in the hierarchy: this ensures that the most informed source is always referenced. The first one and the last two are the most significant. The RB rule (for Rule of Bivalence) corresponds to the introduction rule for the BI operator. When an agent can infer the truth value of a formula ϕ in any state of affairs (i.e. from both ψ and $\neg\psi$), then an agent $j \succeq i$ becomes informed about it. The BI-operator is closed under standard connectives. The I-intro rule infers from an operation of becoming informed from every source strictly higher in the hierarchy a state of being informed about the formula. This rule also plays the role of elimination rule for the BI operator. The I-elim rule makes information shared by all agents into a valid formula for the current agent.

Example 5. Consider the following scenario, with three agents involved in the reasoning process and ordered as $h \preceq i \preceq j$:

$$\dfrac{\dfrac{(\chi \to (\psi \vee \phi))_i \qquad (\neg\chi \to (\psi \vee \phi))_i}{BI_j(\psi \vee \phi)_i} \qquad \dfrac{(\neg\rho \to \neg\phi)_h \qquad (\rho \to \neg\phi)_h}{BI_j\neg\phi_h}}{BI_j\psi_h}$$

We assume that j is at k-depth and she does not hold the truth value of the formula ψ. The derivation of such formula occurs at $k+2$ because in the derivation of $BI_j\psi_h$ two instances of RB involving distinct agents occur, namely the derivation of $(\psi \vee \phi)$ from both χ and $\neg\chi$ for agent i, and the derivation of

$\neg\phi$ from both ρ and $\neg\rho$ for agent h. Hence, the elements of the set of premises of the derivation of ψ are four formulas: $(\chi \to (\psi \vee \phi))_i, (\neg\chi \to (\psi \vee \phi))_i, (\neg\rho \to \neg\phi)_h, (\rho \to \neg\phi)_h$.[6]

Example 6. Consider the following scenario:

$$\dfrac{\dfrac{(\phi \vee \psi)_j \qquad \neg\psi_j}{\phi_j} \qquad \dfrac{\dfrac{\dfrac{(\chi \to \gamma)_i \qquad (\neg\chi \to \gamma)_i}{BI_j\gamma_i} \qquad BI_j\gamma_n}{I_j\gamma_n}\ \forall n \preceq j}{\gamma_j}}{(\phi \wedge \gamma)_j}$$

Let's begin with the longer branch. Agent j (assuming is at $k \geq 0$ depth), does not hold a truth value for formula γ. However, agent i can derive the truth value of γ from both χ and $\neg\chi$ at $k+1$-depth. Hence, $BI_j\gamma_i$ can be inferred. Now suppose that not only i, but every agent $n \preceq j$ (i.e. every agent with more information than j) holds γ, and thus agent j becomes informed about it by all agents higher in the hierarchy. As such, by the I-intro rule, it holds that $I_j\gamma_i$. By the I-elim rule, agent j holds that γ, and hence γ_j holds. Moving to the other branch, agent j holds $(\phi \vee \psi)$. In addition, she holds that $\neg\psi$ is the case. Hence, by the $\vee$-elim2 rules of MA-DBBL$^{\mathcal{C}}$, agent j is able to infer that ϕ is the case, hence ϕ_j holds. Using the $\wedge$-intro1 rule, agent j is able to infer that $(\phi \wedge \gamma)$ is the case, hence $(\phi \wedge \gamma)_j$ holds. This example shows how rules of MA-DBBL$^{\mathcal{C}}$ and of MA-DBBL$^{\mathcal{K}}$ operate harmoniously at two different levels in the same tree: the rules of the first allows the formalization of the operations that an agent performs with her own information, while the second formalizes the operations of becoming informed and being informed.

The number of instances of RB in a derivation executed by distinct agents establishes the depth of the reasoning in which they are (collectively) involved, and f is still an operation to establish the depth of formulas. Then we generalize derivability and refutability for the logic MA-DBBL:

Definition 12 (Derivability for MA-DBBL). *Given a set of formulas Γ_j, formula ϕ_i of $\mathcal{L}^{\mathcal{C}}$, formulas $BI_j\phi_i$ and $I_j\phi$ of $\mathcal{L}^{\mathcal{K}}$, all operations f and $i \preceq j$:*

- $\Gamma_j \vdash_0^f \phi$ *iff* $\Gamma_j \vdash_0 \phi$;
- $\Gamma_j \vdash_{k+1}^f BI_j\phi_i$ *iff*
  - *there are $k+1$ distinct instances of RB such that for each application of the rule there is some formula $\psi_i \in f(\Gamma_i \cup \{\phi_i\})$, such that $\Gamma_i, \psi_i \vdash_k^f \phi_i$ and $\Gamma_i, \neg\psi_i \vdash_k^f \phi_i$;*
  - *or $BI_j\phi_i$ is obtained from $k+1$-depth derivable BI-formulae by an intelim rule of MA-DBBL$^{\mathcal{K}}$;*

[6] We stress here that while in single agent DBBL the depth of the reasoning process is given by *nested* applications of RB by the agent, in MA-DBBL are the instances of RB indexed by *distinct* agents that determine such depth.

– $\Gamma_j \vdash^f_{k+1} I_j\phi_i$ *iff*
  – *there is an instance of an I-introduction rule with n premises of the form* $\Gamma_j \vdash^f_{k+1} BI_j\phi_i$, *for n agents* $i \preceq j$;
  – *or* ϕ_i *is obtained from a* $k+1$*-depth derivable I-formula by an I-elim rule of MA-DBBL*$^{\mathcal{K}}$.

The first case holds trivially. For the second case and $k = 0$ suppose that $\Gamma_j \vdash^f_{k+1} BI_j\phi_i$, but that there is no formula ψ_i such that $\Gamma_i, \psi_i \vdash^f_k \phi_i$ nor $\Gamma_i, \neg\psi_i \vdash^f_k \phi_i$ can be satisfied: then the RB rule from Figure 2 cannot be applied and $\Gamma_j \nvdash^f_{k+1} BI_j\phi_i$ against the hypothesis. For $k > 0$ the previous case requires multiple distinct instances of RB. The second part of the second point follows from the rules of Figure 2. For the third case, suppose that $\Gamma_j \vdash^f_{k+1} I_j\phi_h$ is valid for $h \preceq i \preceq j$. Suppose also that $\Gamma_j \vdash^f_{k+1} BI_j\phi_i$ holds, but $\Gamma_j \vdash^f_{k+1} BI_j\phi_h$ does not. Hence, the I-introduction rule from Figure 2 cannot be applied and thus $\Gamma_j \nvdash^f_{k+1} I_j\phi_i$, against the hypothesis. The second part of the third case follows from the definition of the I-elimination rule of Figure 2.

Definition 13 **(k+1 Inconsistency).** *Γ_i is k+1-depth inconsistent if and only if $\Gamma_i \cup \{\psi_i\}$ and $\Gamma_i \cup \{\neg\psi_i\}$ are both k-depth inconsistent for some $\psi_i \in f(\Gamma_i)$.*

Definition 14 **(MA-DBBL Refutability).** *Given a set of formulas $\Gamma_i \in \mathcal{L}^{\mathcal{CK}}$ and f:*

– *An MA-DBBL derivation is closed when it contains both ϕ_i and $\neg\phi_i$, for some $\phi_i \in f(\Gamma_i)$;*
– *An MA-DBBL refutation of Γ_i is a closed MA-DBBL derivation for Γ_i;*
– *Γ_i is MA-DBBL-refutable if there is a closed MA-DBBL derivation for Γ_i.*

5 Meta-theory

In the present section we provide essential meta-theoretic results.

Theorem 1 **(Soundness).** *If $\Gamma_i \vdash^f_k \phi_j$ then $\Gamma_i \vDash^f_k \phi_j$.*

Proof. The proof proceeds by induction on the depth of the derivability relation:

– for $k = 0$-depth: consider the case $\phi_j \wedge \psi_j$, with ϕ_j, ψ_j in Γ_j. Then, $\Gamma_j \vdash_0 \phi_j$, $\Gamma_j \vdash_0 \psi_j$ and by $\wedge$-intro1 their conjunction is derivable at $k = 0$. There is a matching case in the semantics such that if $1_j, 1_j$ then the conjunction is in the consequence set $Cn_0(\Gamma_j)$. The same holds for all rules of MA-DBBL$^{\mathcal{C}}$.
– for $k > 0$-depth:
  – BI formulae: consider $\Gamma_j \vdash^f_{k+1} BI_j\phi_i$, obtained by an instance of RB. Then $\Gamma_i \vdash^f_k \psi_i \to \phi_i$ and $\Gamma_i \vdash^f_k \neg\psi_i \to \phi_i$ for some $\psi_i \in f(\Gamma_i)$, while $\Gamma_j \nvdash^f_k \phi_j$. By the former $\Gamma_i \vdash^f_k \phi_i$, which grants $\Gamma_i \vDash^f_k \phi_i$, i.e. $v(\phi_i) = 1$; the latter means $\Gamma_j \nvDash^f_k \phi_j$, i.e. $v(\phi_j) = 0$ or $v(\phi_j) = *$; in both cases, according to the evaluation function in Definition 7, if 1_i then $BI_j\phi_i = 1$.

– I formulae: if $\Gamma_j \vdash^f_{k+1} I_j\phi_j$ then $BI_j\phi_i, \forall i \preceq j$. The above argument holds for versions of the RB rules with premises indexed by every $i \preceq j$, with matching case in the truth-table for the I operator.

Theorem 2 (Completeness). *If $\Gamma_j \vDash^f_k \phi_j$ then $\Gamma_j \vdash^f_k \phi_j$.*

Proof. The proof proceeds by induction on the depth of the consequence relation:

- for $k = 0$-depth we follow the proof in [7]: suppose that $\Gamma_j \vDash_0 \phi_j$ and $\Gamma_j \nvdash_0 \phi_j$. Then Γ_j is not 0-depth refutable; otherwise, by definition of 0-depth proof, it should hold that $\Gamma_j \vdash_0 \phi_j$ (as well as $\Gamma_j \vdash^f_0 \neg\phi_j$) against the hypothesis. Now, consider the set $\Gamma^0_j = \{\psi_j \mid \Gamma_j \vdash_0 \psi_j\}$; since Γ_j is not 0-depth refutable, there are no formulas χ_j, and $\neg\chi_j$ such that they are both in Γ^0_j. Then, the function $\mathcal{V}$, defined as follows:

$$\mathcal{V}(\chi_j) = \begin{cases} 1 \text{ if } \chi_j \in \Gamma^0_j \\ 0 \text{ if } \neg\chi_j \in \Gamma^0_j \\ * \text{ otherwise} \end{cases} \tag{1}$$

  is a three-value valuation that agrees with Table 1. Consider now the following formula case: $\mathcal{V}(\chi_j) = \mathcal{V}(\rho_j) = *$; then $\neg(\chi \vee \rho)_j \notin \Gamma^0_j$. Otherwise, by definition of Γ^0_j and by the $\vee$-elim3 rule of Figure 1, $\neg\chi_j$ and $\neg\rho_j$ should also be in Γ^0_j. Therefore, by definition of $\mathcal{V}$, $\mathcal{V}(\chi_j) = \mathcal{V}(\rho_j) = 0$, against our assumption. Hence $\mathcal{V}(\chi \vee \rho)_j \neq 0$. Moreover, $(\chi \vee \rho)_j$, may or may not belong to Γ^0_j, and so either $\mathcal{V}(\chi \wedge \rho)_j = 1$ or $\mathcal{V}(\chi \wedge \rho)_j = *$. Finally:
  i) $\psi_j \in \Gamma^0_j$, for all $\psi_j \in \Gamma_j$ and so, by definition of $\mathcal{V}$, $\mathcal{V}$ satisfies all $\psi_j \in \Gamma_j$;
  ii) by the hypothesis that $\Gamma_j \nvdash^f_0 \phi_j$, $\phi_j \notin \Gamma^0_j$ and so $\mathcal{V}$ does not satisfy ϕ_j. Hence $\Gamma_j \nvDash^f_0 \phi_j$, against the initial assumption.
- for $k > 0$-depth; we denote, as above, with Γ^k_i the theoremhood set of Γ_i and depth k. Functions are defined for the theoremhood sets of BI and I formulas:

$$\mathcal{V}(BI_j\chi_i) = \begin{cases} 1 \text{ if } \chi_i \in \Gamma^k_i, \chi_j \notin \Gamma^k_j \text{ and } \neg\chi_j \notin \Gamma^k_j \\ 0 \text{ if } \neg\chi_i \in \Gamma^k_i, \chi_j \notin \Gamma^k_j \text{ and } \neg\chi_j \notin \Gamma^k_j \\ * \text{ if } \chi_i \notin \Gamma^k_i \text{ and } \neg\chi_i \notin \Gamma^k_i \end{cases} \tag{2}$$

$$\mathcal{V}(I_j\chi) = \begin{cases} 1 \text{ if } \chi_i \in \Gamma^k_i \text{ for all } i \preceq j \in \mathcal{S}, \text{ and } \chi_i \in \Gamma^{k+1}_j \\ 0 \text{ if } \neg\chi_i \in \Gamma^k_i \text{ for all } i \preceq j \in \mathcal{S}, \text{ and } \neg\chi_i \in \Gamma^{k+1}_j \\ * \text{ if } \chi_i \notin \Gamma^k_i \text{ and } \neg\chi_i \notin \Gamma^k_i \text{ for all } i \preceq j \in \mathcal{S}. \end{cases} \tag{3}$$

  – For BI formulas: Suppose that $\Gamma_j \vDash^f_{k+1} BI_j\phi_i$ and $\Gamma_j \nvdash^f_{k+1} BI_j\phi_i$. Then Γ_j is not $k+1$-depth refutable; otherwise, by definition of $k+1$-depth proof, it should hold that $\Gamma_i \vdash^f_k \phi_i$ (as well as $\Gamma_i \vdash^f_k \neg\phi_i$), for some $\phi \in f(\Gamma_i)$; hence $\Gamma_j \vdash^f_{k+1} BI_j\phi_i$ is derivable by RB, against the hypothesis. Now, consider the set $\Gamma^{k+1}_j = \{BI_j\phi_i \mid \Gamma_j \vdash^f_{k+1} BI_j\phi_i\}$. Since Γ_j is not $k+1$-depth refutable,

there is no formula such that $BI_j\chi_i$ and $BI_j\neg\chi_i$ are both in Γ_j^{k+1}. Then, consider $\mathcal{V}(BI_j\phi_i) = 1$. Then $\phi_i \in \Gamma_i^k$, and both ϕ_j and $\neg\phi_j \notin \Gamma_j^k$. Otherwise $\mathcal{V}(\phi_i) = 0$ and $\mathcal{V}(\phi_j) = 1$ or $\mathcal{V}(\phi_j) = 0$:
–– if $\mathcal{V}(\phi_i) = 0$, the premises of the relevant RB cannot be satisfied, and $\mathcal{V}(BI_j\phi_i) \neq 1$ against our assumption;
–– if $\mathcal{V}(\phi_j) = 1$, by 0-depth evaluation $\phi_j \in \Gamma_j^k$ which is against the definition of $\mathcal{V}(BI_j\phi_i)$ from the assumption $\Gamma_j \vDash_{k+1}^f BI_j\phi_i$;
–– and if $\mathcal{V}(\phi_j) = 0$, by 0-depth evaluation $\neg\phi_j \in \Gamma_j^k$, as above against the definition of $\mathcal{V}(BI_j\phi_i)$ from the assumption $\Gamma_j \vDash_{k+1}^f BI_j\phi_i$;
– For I-formulas: Consider $\Gamma_j^{k+1} = \{I_j\phi_i \mid \Gamma_j \vdash_{k+1}^f I_j\phi_i\}$ and $\mathcal{V}(I_j\phi_i) = 1$. Then, $\phi_i \in \Gamma_i^k$ for any $i \preceq j \in \mathcal{S}$. Then, by the definition of I, $\phi_j \in \Gamma_j^{k+1}$. Otherwise $\phi_i \notin \Gamma_i^k$ for at least one $i \preceq j \in \mathcal{S}$. But then $\phi_j \notin \Gamma_j^{k+1}$ and hence $\Gamma_j \nvDash_{k+1}^f I_j\phi_i$, against the initial assumption.

6 Conclusions and Future Work

MA-DBBL is an extension of Depth Bound Boolean Logic with formulas indexed by multiple agents. It simulates reasoning by agents ordered on their cognitive abilities, in terms of inferential power based on data access. The language includes a one-to-one information transmission operator between source and receiver which resembles private announcements; and an operator induced by private communications between every source and a given receiver, which is equivalent to everybody knows. Such operations allow agents with poorer cognitive capacities (inferential abilities and data access) to become informed of content otherwise available only to better equipped agents. This feature is especially well fitted for human-machine interaction settings, where inferential abilities and data access among agents varies; or in distributed systems, where agents with different inferential-bounds work towards a common goal.

Future extensions of this logic include: an appropriate relational semantics; a non-monotonic extension, through updates with contradictory information which is essential in human-machine interaction contexts; the definition of trust on agents and the information they hold, and the update of the order of sources based on previous trust evaluations including negative trust, see e.g. [17]; characterization of resource bounds by considering access to information as costly, i.e. leading to a consumption of resources, see e.g. [18].

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$$\frac{(\psi \to \phi)_i \qquad (\neg\psi \to \phi)_i}{BI_j\phi_i}\,\mathrm{RB}$$

$$\frac{BI_j\ \psi_h \qquad BI_j\ \phi_i}{BI_j\ (\phi \wedge \psi)_h} \qquad \frac{BI_j\ \neg\phi_i}{BI_j\ \neg(\phi \wedge \psi)_i} \qquad \frac{BI_j\ \neg\psi_i}{BI_j\ \neg(\phi \wedge \psi)_i}$$

$$\frac{BI_j\ \phi_i}{BI_j\ (\phi \vee \psi)_i} \qquad \frac{BI_j\ \psi_i}{BI_j\ (\phi \vee \psi)_i} \qquad \frac{BI_j\ \neg\phi_h \qquad BI_j\ \neg\psi_i}{BI_j\ \neg(\phi \vee \psi)_h}$$

$$\frac{BI_j\ \neg\phi_i}{BI_j\ (\phi \to \psi)_i} \qquad \frac{BI_j\ \psi_i}{BI_j\ (\phi \to \psi)_i} \qquad \frac{BI_j\phi_h \qquad BI_j\neg\psi_i}{BI_j\ \neg(\phi \to \psi)_h}$$

$$\frac{BI_j\ (\phi \wedge \psi)_i}{BI_j\ \phi_i} \qquad \frac{BI_j\ (\phi \wedge \psi)_i}{BI_j\ \psi_i}$$

$$\frac{BI_j\ \neg(\phi \wedge \psi)_h \qquad BI_j\ \phi_i}{BI_j\ \neg\psi_h} \qquad \frac{BI_j\ \neg(\phi \wedge \psi)_h \qquad BI_j\ \psi_i}{BI_j\ \neg\phi_h}$$

$$\frac{BI_j\ (\phi \vee \psi)_h \qquad \neg BI_j\ \phi_i}{BI_j\ \psi_h} \qquad \frac{BI_j\ (\phi \vee \psi)_h \qquad \neg BI_j\ \psi_i}{BI_j\ \phi_h}$$

$$\frac{BI_j\neg(\phi \vee \psi)_i}{BI_j\neg\ \phi_i} \qquad \frac{BI_j\neg(\phi \vee \psi)_i}{BI_j\neg\ \psi_i}$$

$$\frac{BI_j\ (\phi \to \psi)_h \qquad BI_j\ \phi_i}{BI_j\ \psi_h} \qquad \frac{BI_j\ (\phi \to \psi)_h \qquad BI_j\ \neg\psi_i}{BI_j\ \neg\phi_h}$$

$$\frac{BI_j\ \neg(\phi \to \psi)_i}{BI_j\ \neg\psi_i} \qquad \frac{BI_j\ \neg(\phi \to \psi)_i}{BI_j\ \phi_i}$$

$$\frac{BI_j\phi_i,\ \forall i \preceq j \in \mathcal{S}}{I_j\phi_i}\ I\text{-intro} \quad \frac{I_j\phi_i}{\phi_j}\ I\text{-elim}$$

Fig. 2: Introduction and Elimination Rules for MA-DBBL$^{\mathcal{K}}$ with $h \preceq i \preceq j$